

EXACT SOLUTIONS OF SECOND ORDER OBSTACLE PROBLEMS USING REDUCTION-TO-FIRST ORDER METHOD

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ABSTRACT: In this article we used the Reduction-to-First –Order method to solve the systems of second order boundary value problems. Two examples are presented to get exact solution, which illustrate the effectiveness of the method.

Key Words: Reduction-to-First-Order, Obstacle Problems, System of Boundary value Problems.

1. INTRODUCTION

Reduction to first order (RFO) method is presented to find the exact solutions of second order obstacle, unilateral and contact problems described in its general form as:

$$u''(x) = \begin{cases} f(x) & a \leq x < c, \\ g(x)u(x) + f(x) + r & c \leq x \leq d, \\ f(x) & d < x \leq b, \end{cases} \quad (1)$$

With the boundary conditions, $u(a) = \alpha_1$ and $u(b) = \alpha_2$, where α_1, α_2 are constants. The continuity condition of $u(x)$ and $u'(x)$ at c and d are also applied. Here $f(x)$ and $g(x)$ are the continues functions on $[a, b]$ and $[c, d]$ respectively.

Generally it is impossible to acquire the analytical form of the solution of for the arbitrary choice of $f(x)$ and $g(x)$. For this purpose some numerical methods are opted in recent years to get approximate solutions of the problems of the kind (1). Such type of systems arise in the study of obstacle, unilateral and contact boundary value problems and have important applications in other branches of pure and applied sciences.

In 1987, M. A. Noor and A. K. Khalifa solved the unilateral problems using cubic spline method in [1]. M. A. Noor and S. I. A. Tirmizi used the technique of finite difference to solve obstacle problems [2] in 1988. E. A. Al-Said used spline [3], smooth spline [4] and cubic spline [5] methods to find the numerical solutions of system of system of second order boundary value problems in 1996, 1999 and 2001 respectively. The same second order problems has been solved using a parametric cubic spline approach [6] in 2003 by A. Khan and T. Aziz. In 2010, the solution of same second order system of boundary value problems was solved numerically using Galerkin’s finite element formulation by S. Iqbal et al [7], also S. Iqbal solved the second order obstacle problem using cubic Lagrange polynomial in Galerkin’s finite element fashion [8]. The numerical solutions of obstacle problems of second order have been given by G. B. Loghmani et al using B-spline function [9] in 2011.

A spatially Adaptive Grid Refinement Scheme was used for the Finite Element solution of a Second order Obstacle Problem in [10] by S. Iqbal et al in 2013.

Reduction-to-First-Order Method (RFO) was introduced by L. E. Nicholas Afima [11] in 1991. In 2013, RFO was presented first time at any forum in a workshop held at 11th Conference on Frontiers of Information Technology by A. M. Siddiqui and T. Haroon [12]. Reduction-to-First-Order Method requires a suitable transformation to reduce the order of the second order linear nonhomogeneous differential equations with constant coefficients. These reduced order differential equations can then be solved using the integrating factor technique. To illustrate the general utility of the method two

examples of second order obstacle problem are presented. This method is also applicable to linear equations of higher order with constant coefficients.

2. REDUCTION TO FIRST ORDER (AN ALGORITHM)

The formulation of the working algorithm of the Reduction –to-First-Order Method can be expressed in the following way [11], [12]:

(a) Write the governing differential equation

$$u'' + mu' + nu = h(x) \quad (2)$$

(b) Factorize equation (2) as

$$(u' - \alpha u)' - \beta(u' - \alpha u) = h(x) \quad (3)$$

and the equating of coefficient shows that $\alpha + \beta = -m$ and $\alpha\beta = n$, so that α and β are the roots of the quadratic equation $\lambda^2 + m\lambda + n = 0$.

(c) Introduce the new variable

$$z = u' - \alpha u \quad (4)$$

So that, equation (2) will reduce to the first order linear differential equation as

$$z' - \beta z = h(x) \quad (5)$$

With integration factor $e^{-\beta x}$

(d) Solve equation (5) for z

$$z = Ce^{\beta x} + e^{\beta x} \int e^{-\beta x} h(x) dx \quad (6)$$

(e) Restore the original variable u by equation (4), once again first order linear differential equation will appear, which will be solved by method of integrating factor to find the solution of equation (2).

3. ILLUSTRATIVE EXAMPLES TO DEMONSTRATE REDUCTION TO FIRST ORDER

In this section the solution of two second order obstacle problems are given using Reduction-to-First-Order method.

EXAMPLE 3.1. We consider the second order obstacle problem of the following form

$$u'' = \begin{cases} 0 & \text{for } 0 \leq x < \frac{\pi}{4} \text{ and } \frac{3\pi}{4} < x \leq \pi \\ u - 1 & \text{for } \frac{\pi}{4} \leq x \leq \frac{3\pi}{4} \end{cases} \quad (7)$$

With the boundary conditions $u(0) = u(\pi) = 0$.

The analytical solution for the above mentioned example is given by [1]-[10]:

$$u = \begin{cases} \frac{4x}{\pi + 4\coth\frac{\pi}{4}} & \text{for } 0 \leq x < \frac{\pi}{4} \\ 1 - \frac{4\cosh(\frac{\pi-x}{2})}{\pi\sinh\frac{\pi}{4} + 4\cosh\frac{\pi}{4}} & \text{for } \frac{\pi}{4} \leq x \leq \frac{3\pi}{4} \\ \frac{4(\pi-x)}{\pi + 4\coth\frac{\pi}{4}} & \text{for } \frac{3\pi}{4} < x \leq \pi \end{cases} \quad (8)$$

The problem is divided in to three cases,

Case1 ($0 \leq x < \frac{\pi}{4}$)

In this case, we have the following differential equation with boundary conditions,

$$u''(x) = 0, u(0) = 0, u\left(\frac{\pi}{4}\right) = c \tag{9}$$

Where c is constant. After solving equation (9), we have

$$u_{\left[0, \frac{\pi}{4}\right]}(x) = \frac{4cx}{\pi} \tag{10}$$

Case2 ($\frac{\pi}{4} \leq x \leq \frac{3\pi}{4}$)

In this case, we have the following differential equation with boundary conditions,

$$u''(x) - u(x) = -1, u\left(\frac{\pi}{4}\right) = c, u\left(\frac{3\pi}{4}\right) = b \tag{11}$$

The equation (11) is rearranged as

$$u''(x) - u'(x) + u'(x) - u(x) = -1 \tag{12}$$

$$\left(u'(x) - u(x)\right)' + u'(x) - u(x) = -1 \tag{13}$$

A new variable z is introduced such that

$$z = u'(x) - u(x)$$

so equation (11) become

$$z'(x) + z(x) = -1 \tag{14}$$

which is first order linear differential equation.

Now, by using integrating factor (IF) method, the solution is

$$z(x) = -1 + De^{-x} \tag{15}$$

Where D is constant of integration.

Restoring the original variable u , the differential equation became,

$$u'(x) - u(x) = -1 + De^{-x} \tag{16}$$

Using IF method the obtained solution is

$$u(x) = 1 - \frac{D}{2}e^{-x} + Ee^x \tag{17}$$

Where E is constant of intation.

The boundary conditions are applied to obtain the value of D and E ,

$$D = -\frac{2e^{\frac{3\pi}{4}}(1-b-e^{\frac{\pi}{2}}+ce^{\frac{\pi}{2}})}{-1+e^\pi} \tag{18}$$

$$E = \frac{e^{-\frac{\pi}{4}}(1-c-e^{\frac{\pi}{2}}+be^{\frac{\pi}{2}})}{-1+e^\pi} \tag{19}$$

Putting the values of D and E , the solution is,

$$u_{\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]}(x) = 1 + \frac{e^{\frac{3\pi}{4}-x}(1-b-e^{\frac{\pi}{2}}+ce^{\frac{\pi}{2}})}{-1+e^\pi} + \frac{e^{-\frac{\pi}{4}+x}(1-c-e^{\frac{\pi}{2}}+be^{\frac{\pi}{2}})}{-1+e^\pi} \tag{20}$$

Case3 ($\frac{3\pi}{4} < x \leq \pi$)

As in Case1, we have the following differential equation with boundary conditions

$$u''(x) = 0, u\left(\frac{3\pi}{4}\right) = b, u(\pi) = 0 \tag{21}$$

After solving equation (21), we have

$$u_{\left[\frac{3\pi}{4}, \pi\right]}(x) = \frac{4b(\pi-x)}{\pi} \tag{22}$$

Hence

$$u(x) = \begin{cases} \frac{4cx}{\pi} & \text{for } 0 \leq x < \frac{\pi}{4} \\ 1 + \frac{e^{\frac{3\pi}{4}-x}(1-b-e^{\frac{\pi}{2}}+ce^{\frac{\pi}{2}})}{-1+e^\pi} + \frac{e^{-\frac{\pi}{4}+x}(1-c-e^{\frac{\pi}{2}}+be^{\frac{\pi}{2}})}{-1+e^\pi} & \text{for } \frac{\pi}{4} \leq x \leq \frac{3\pi}{4} \\ \frac{4b(\pi-x)}{\pi} & \text{for } \frac{3\pi}{4} < x \leq \pi \end{cases} \tag{23}$$

Now, to obtain continuous solution for the given obstacle problem, following continuity conditions are used to find the values of b and c .

$$\lim_{x \rightarrow \frac{\pi}{4}^-} u'(x) = \lim_{x \rightarrow \frac{\pi}{4}^+} u'(x) \tag{24}$$

$$\lim_{x \rightarrow \frac{3\pi}{4}^-} u'(x) = \lim_{x \rightarrow \frac{3\pi}{4}^+} u'(x) \tag{25}$$

We get

$$b = c = \frac{(-1+e^{\frac{\pi}{2}})\pi}{4+4e^{\frac{\pi}{2}}-\pi+\pi e^{\frac{\pi}{2}}} \tag{26}$$

By substituting values of b and c in equation (23).

We have,

$$u(x) = \begin{cases} \frac{4(-1+e^{\frac{\pi}{2}})x}{4+4e^{\frac{\pi}{2}}-\pi+\pi e^{\frac{\pi}{2}}} & \text{for } 0 \leq x < \frac{\pi}{4} \\ \frac{e^{-\frac{\pi}{4}-x}(-4e^{\frac{\pi}{2}}-4e^{2x}-e^{\frac{\pi}{4}+x}(-4+\pi)+e^{\frac{3\pi}{4}+x}(4+\pi))}{4-\pi+e^{\frac{\pi}{2}}(4+\pi)} & \text{for } \frac{\pi}{4} \leq x \leq \frac{3\pi}{4} \\ \frac{4(-1+e^{\frac{\pi}{2}})(\pi-x)}{4+4e^{\frac{\pi}{2}}-\pi+\pi e^{\frac{\pi}{2}}} & \text{for } \frac{3\pi}{4} < x \leq \pi \end{cases} \tag{27}$$

After simplification, we get the exact solution as presented in equation (8).

$$u(x) = \begin{cases} \frac{4(e^{\frac{\pi}{4}}-e^{-\frac{\pi}{4}})x}{4e^{-\frac{\pi}{4}}+4e^{\frac{\pi}{4}}-\pi e^{-\frac{\pi}{4}}+\pi e^{\frac{\pi}{4}}} & \text{for } 0 \leq x < \frac{\pi}{4} \\ \frac{(4-\pi)+(4+\pi)e^{\frac{\pi}{2}}-4e^{\frac{3\pi}{4}-x}-4e^{-\frac{\pi}{4}+x}}{4-\pi+e^{\frac{\pi}{2}}(4+\pi)} & \text{for } \frac{\pi}{4} \leq x \leq \frac{3\pi}{4} \\ \frac{4(e^{\frac{\pi}{4}}-e^{-\frac{\pi}{4}})(\pi-x)}{4e^{-\frac{\pi}{4}}+4e^{\frac{\pi}{4}}-\pi e^{-\frac{\pi}{4}}+\pi e^{\frac{\pi}{4}}} & \text{for } \frac{3\pi}{4} < x \leq \pi \end{cases} \tag{28}$$

$$u(x) = \begin{cases} \frac{4x}{\pi+4\left(\frac{e^{\frac{\pi}{4}}+e^{-\frac{\pi}{4}}}{e^{\frac{\pi}{4}}-e^{-\frac{\pi}{4}}}\right)} & \text{for } 0 \leq x < \frac{\pi}{4} \\ 1 - \frac{2e^{\frac{\pi}{2}-x}+2e^{\frac{\pi}{2}+x}}{\frac{\pi}{2}(e^{\frac{\pi}{4}}-e^{-\frac{\pi}{4}})+2(e^{\frac{\pi}{4}}+e^{-\frac{\pi}{4}})} & \text{for } \frac{\pi}{4} \leq x \leq \frac{3\pi}{4} \\ \frac{4(\pi-x)}{\pi+4\left(\frac{e^{\frac{\pi}{4}}+e^{-\frac{\pi}{4}}}{e^{\frac{\pi}{4}}-e^{-\frac{\pi}{4}}}\right)} & \text{for } \frac{3\pi}{4} < x \leq \pi \end{cases} \tag{29}$$

Since $\sinh\theta = \frac{e^\theta - e^{-\theta}}{2}$ and $\cosh\theta = \frac{e^\theta + e^{-\theta}}{2}$.

EXAMPLE 3.2. Here we consider a second example of system of boundary value problem of the following form:

$$u'' = \begin{cases} 2 & \text{for } 0 \leq x < \frac{\pi}{4} \text{ and } \frac{3\pi}{4} < x \leq \pi \\ u + 1 & \text{for } \frac{\pi}{4} \leq x \leq \frac{3\pi}{4} \end{cases} \tag{30}$$

With boundary conditions $u(0) = u(\pi) = 0$. The analytical solution for the above mentioned example is given by [9].

$$u(x) = \begin{cases} x^2 + \left(\frac{(\pi^2-16)\sinh\frac{\pi}{4}}{4\pi\sinh\frac{\pi}{4}+16\cosh\frac{\pi}{4}} - \frac{\pi}{2}\right)x & \text{for } 0 \leq x < \frac{\pi}{4} \\ -1 - \frac{(\pi^2-16)\cosh\left(\frac{\pi}{2}-x\right)}{4\pi\sinh\frac{\pi}{4}+16\cosh\frac{\pi}{4}} & \text{for } \frac{\pi}{4} \leq x \leq \frac{3\pi}{4} \\ x^2 + \left(\frac{(\pi^2-16)\sinh\frac{\pi}{4}}{4\pi\sinh\frac{\pi}{4}+16\cosh\frac{\pi}{4}}\right)(\pi-x) + \frac{\pi}{2}(\pi-3x) & \text{for } \frac{3\pi}{4} < x \leq \pi \end{cases} \tag{31}$$

The problem is divided into three cases,

Case1 ($0 \leq x < \frac{\pi}{4}$)

In this case, we have the following differential equation with boundary conditions,

$$u''(x) = 2, u(0) = 0, u\left(\frac{\pi}{4}\right) = c \tag{32}$$

Where c is constant. After solving equation (32), we have

$$u_{[0, \frac{\pi}{4}]}(x) = \frac{4cx}{\pi} - \frac{\pi x}{4} + x^2 \tag{33}$$

Case2 ($\frac{\pi}{4} \leq x \leq \frac{3\pi}{4}$)

In this case, we have the following differential equation with boundary conditions,

$$u''(x) - u(x) = 1, u(\frac{\pi}{4}) = c, u(\frac{3\pi}{4}) = b \tag{34}$$

The equation (34) is rearranged as

$$u''(x) - u'(x) + u'(x) - u(x) = 1 \tag{35}$$

$$(u'(x) - u(x))' + u'(x) - u(x) = 1 \tag{36}$$

A new variable z is introduced such that

$$z = u'(x) - u(x)$$

so the equation (34) become

$$z'(x) + z(x) = 1 \tag{37}$$

which is first order linear differential equation.

Now, by using integrating factor (IF) method, the solution is

$$z(x) = 1 + De^{-x}$$

(38)

Where D is constant of integration.

Restoring the original variable u , the differential equation became,

$$u'(x) - u(x) = 1 + De^{-x} \tag{39}$$

Using IF method the obtained solution is

$$u(x) = -1 - \frac{D}{2}e^{-x} + Ee^x \tag{40}$$

Where E is constant of integration.

The boundary conditions are applied to obtain the value of D and E ,

$$D = -\frac{2e^{\frac{3\pi}{4}}(-1-b+e^{\frac{\pi}{2}}+ce^{\frac{\pi}{2}})}{-1+e^\pi} \tag{41}$$

$$E = \frac{e^{-\frac{\pi}{4}}(-1-c+e^{\frac{\pi}{2}}+be^{\frac{\pi}{2}})}{-1+e^\pi} \tag{42}$$

Putting the values of D and E , the solution is,

$$u_{[\frac{\pi}{4}, \frac{3\pi}{4}]}(x) = -1 + \frac{e^{\frac{3\pi}{4}-x}(-1-b+e^{\frac{\pi}{2}}+ce^{\frac{\pi}{2}})}{-1+e^\pi} + \frac{e^{-\frac{\pi}{4}+x}(-1-c+e^{\frac{\pi}{2}}+be^{\frac{\pi}{2}})}{-1+e^\pi} \tag{43}$$

Case3 ($\frac{3\pi}{4} < x \leq \pi$)

As in Case1, we have the following differential equation with boundary conditions

$$u''(x) = 2, u(\frac{3\pi}{4}) = b, u(\pi) = 0 \tag{44}$$

After solving equation (44), we have

$$u_{[\frac{3\pi}{4}, \pi]}(x) = \frac{(16b+\pi(3\pi-4x))(\pi-x)}{4\pi} \tag{45}$$

Hence

$$u(x) = \begin{cases} \frac{4cx}{\pi} - \frac{\pi x}{4} + x^2 & \text{for } 0 \leq x < \frac{\pi}{4} \\ -1 + \frac{e^{\frac{3\pi}{4}-x}(-1-b+e^{\frac{\pi}{2}}+ce^{\frac{\pi}{2}})}{-1+e^\pi} + \frac{e^{-\frac{\pi}{4}+x}(-1-c+e^{\frac{\pi}{2}}+be^{\frac{\pi}{2}})}{-1+e^\pi} & \text{for } \frac{\pi}{4} \leq x \leq \frac{3\pi}{4} \\ \frac{(16b+\pi(3\pi-4x))(\pi-x)}{4\pi} & \text{for } \frac{3\pi}{4} < x \leq \pi \end{cases} \tag{46}$$

Now, to obtain continuous solution for the given obstacle problem, following continuity conditions are used to find the values of b and c .

$$\lim_{x \rightarrow \frac{\pi}{4}^-} u'(x) = \lim_{x \rightarrow \frac{\pi}{4}^+} u'(x) \tag{47}$$

$$\lim_{x \rightarrow \frac{3\pi}{4}^-} u'(x) = \lim_{x \rightarrow \frac{3\pi}{4}^+} u'(x) \tag{48}$$

We get

$$b = c = -\frac{\pi(-4+4e^{\frac{\pi}{2}}+\pi+\pi e^{\frac{\pi}{2}})}{4(4+4e^{\frac{\pi}{2}}-\pi+\pi e^{\frac{\pi}{2}})} \tag{49}$$

By substituting values of b and c in equation (46). we have,

$$u(x) = \begin{cases} \frac{1}{4}x\left(-4-\pi-\frac{8(-4+\pi)}{4-\pi+e^{\frac{\pi}{2}}(4+\pi)}+4x\right) & \text{for } 0 \leq x < \frac{\pi}{4} \\ \frac{e^{-\frac{\pi}{4}-x}\left(4e^{\frac{3\pi}{4}+x}(-4+\pi)-4e^{\frac{3\pi}{4}+x}(4+\pi)-e^{\pi(-16+\pi^2)}-e^{2x(-16+\pi^2)}\right)}{4(4-\pi+e^{\frac{\pi}{2}}(4+\pi))} & \text{for } \frac{\pi}{4} \leq x \leq \frac{3\pi}{4} \\ \frac{\left(\frac{4\pi(-4+4e^{\frac{\pi}{2}}+\pi+\pi e^{\frac{\pi}{2}})}{4+4e^{\frac{\pi}{2}}-\pi+\pi e^{\frac{\pi}{2}}}+\pi(3\pi-4x)\right)(\pi-x)}{4\pi} & \text{for } \frac{3\pi}{4} < x \leq \pi \end{cases} \tag{50}$$

After simplification, we get the exact solution as in equation (8).

$$u(x) = \begin{cases} x^2 + \left(-1 - \frac{\pi}{4} - \frac{2(-4+\pi)}{4-\pi+e^{\frac{\pi}{2}}(4+\pi)}\right)x & \text{for } 0 \leq x < \frac{\pi}{4} \\ -1 - \frac{(-16+\pi^2)\text{Cosh}\left(\frac{1}{2}(\pi-2x)\right)}{4(4\text{Cosh}\frac{\pi}{4}+\pi\text{Sinh}\frac{\pi}{4})} & \text{for } \frac{\pi}{4} \leq x \leq \frac{3\pi}{4} \\ x^2 + \left(\frac{-3\pi}{2} - \frac{(\pi^2-16)(e^{\frac{\pi}{2}}-1)}{4\pi(e^{\frac{\pi}{2}}-1)+16(e^{\frac{\pi}{2}}+1)}\right)x + \frac{(\pi^3-16\pi)(e^{\frac{\pi}{2}}-1)}{4\pi(e^{\frac{\pi}{2}}-1)+16(e^{\frac{\pi}{2}}+1)} + \frac{\pi^2}{2} & \text{for } \frac{3\pi}{4} < x \leq \pi \end{cases}$$

$$u(x) = \begin{cases} x^2 + \left(\frac{(\pi^2-16)(e^{\frac{\pi}{4}}-e^{-\frac{\pi}{4}})}{4\pi(e^{\frac{\pi}{4}}-e^{-\frac{\pi}{4}})+16(e^{\frac{\pi}{4}}+e^{-\frac{\pi}{4}})}\right)x & \text{for } 0 \leq x < \frac{\pi}{4} \\ -1 - \frac{(-16+\pi^2)\text{Cosh}\left(\frac{1}{2}(\pi-2x)\right)}{4(4\text{Cosh}\frac{\pi}{4}+\pi\text{Sinh}\frac{\pi}{4})} & \text{for } \frac{\pi}{4} \leq x \leq \frac{3\pi}{4} \\ x^2 + \left(\frac{(\pi^2-16)(e^{\frac{\pi}{4}}-e^{-\frac{\pi}{4}})}{4\pi(e^{\frac{\pi}{4}}-e^{-\frac{\pi}{4}})+16(e^{\frac{\pi}{4}}+e^{-\frac{\pi}{4}})}\right)(\pi-x) + \frac{\pi}{2}(\pi-3x) & \text{for } \frac{3\pi}{4} < x \leq \pi \end{cases}$$

4. CONCLUSION

In this article, Reduction-to-First-Order Method has been presented to solve the systems of second order boundary value problems. Two examples of second order obstacle, unilateral and contact problems demonstrated the exact solutions. RFO always gives the solution using the first order linear method and for higher order differential equations no other special treatment is required.

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